ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE EXISTENCE OF DECAYING SOLUTIONS OF THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR A SEMISTRIP

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In connection with the construction of a more precise theory for the bending of plates [1] there arises the problem of determining the conditions for the existence of decaying solutions for a semistrip free of stresses along the longutudinal edges under various boundary conditions. Sufficient conditions for existence of decaying solutions, which are expressible by means of series in Papkovich functions [3], were obtained in [2] for two problems that correspond to prescribing, on the edge, one condition for the stress and one condition of displacement. The case when both components of displacement are given on the edge, has not been investigated as yet.

In the present paper the Laplace transform is used for the derivation of solutions of the Lamé's equations. This permits one to approach the problems corresponding to different boundary conditions from one viewpoint only, and to determine necessary and sufficient conditions for the existence of decaying solutions.

1. Let us consider four types of conditions on the boundary $x = 0$ of the semistrip

$$\sigma_x(0, y) = f_1(y), \qquad \tau_{xy}(0, y) = f_2(y) \qquad \text{(Problem 1)} \tag{1.1}$$

$$2\mu\, u(0, y) = f_1(y), \qquad \tau_{xy}(0, y) = f_2(y) \qquad \text{(Problem 2)} \tag{1.2}$$

$$\sigma_x(0, y) = f_1(y), \qquad 2\mu\, v(0, y) = f_2(y) \qquad \text{(Problem 3)} \tag{1.3}$$

$$2\mu\, u(0, y) = f_1(y), \qquad 2\mu\, v(0, y) = f_2(y) \qquad \text{(Problem 4)} \tag{1.4}$$

The boundary conditions for $y = \pm 1$, have the following form for each of the four problems:

$$\sigma_y(x, \pm 1) = 0, \qquad \tau_{xy}(x, \pm 1) = 0 \tag{1.5}$$

We shall determine the conditions (necessary and sufficient) which are imposed on the boundary functions $f_1(y)$, $f_2(y)$ in order that the solution of the equations of Lamé

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x\,\partial y} = 0$$
$$\mu\frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x\,\partial y} = 0 \tag{1.6}$$

which corresponds to the prescribed conditions on the boundary and on the semistrip egdes, and has the decaying character in the x-direction, i.e. $u(x, y) \to 0$, $v(x, y) \to 0$ as $x \to \infty$.

Let us consider the solution of Lamé's equations in the class of functions which includes decaying and increasing functions. Let us assume that the order of the increasing functions is nor higher than a power of x as $x \to \infty$.

We now apply Laplace's transform to x in Lamé's equation. Setting

$$U(p; y) = \int_0^\infty u(x, y) e^{-px} dx, \qquad V(p; y) = \int_0^\infty v(x, y) e^{-px} dx \tag{1.7}$$

we obtain for $U(p; y)$ and $V(p; y)$ the following second order nonhomogeneous system of ordinary differential equations

$$\begin{aligned} \frac{\partial^2 U}{\partial y^2} + \frac{\lambda + 2\mu}{\mu} p^2 U + \frac{\lambda + \mu}{\mu} p \frac{\partial V}{\partial y} &= \frac{1}{\mu} \Phi(p, y) \\ \frac{\partial^2 V}{\partial y^2} + \frac{\mu}{\lambda + 2\mu} p^2 V + \frac{\lambda + \mu}{\lambda + 2\mu} p \frac{\partial U}{\partial y} &= \frac{1}{\lambda + 2\mu} \Psi(p, y) \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} \Phi(p, y) &= (\lambda + 2\mu) \frac{\partial u}{\partial x}\Big|_{x=0} + (\lambda + \mu) \frac{\partial v}{\partial y}\Big|_{x=0} + (\lambda + 2\mu) pu(0, y) \\ \Psi(p, y) &= \mu \frac{\partial v}{\partial x}\Big|_{x=0} + (\lambda + \mu) \frac{\partial u}{\partial y}\Big|_{x=0} + \mu pv(0, y) \end{aligned} \tag{1.9}$$

The general solution of (1.8) contains four arbitrary constants depending on p, and it has the form

$$\begin{aligned} U(p, y) &= a_1(p) \sin py + a_2(p) \cos py + a_3(p) py \cos py + \\ &\quad + a_4(p) py \sin py + U_0(p; y) \\ V(p, y) &= (-a_2(p) - \varkappa_1 a_4(p)) \sin py + (a_1(p) - \varkappa_1 a_3(p)) \cos py + \\ &\quad + a_4(p) py \cos py - a_3(p) py \sin py + V_0(p, y) \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} U_0(p, y) &= b_1(p, y) \sin py + b_2(p, y) \cos py + b_3(p, y) py \cos py + \\ &\quad + b_4(p, y) py \sin py \end{aligned} \tag{1.11}$$

$$\begin{aligned} V_0(p, y) &= (-b_2(p, y) - \varkappa_1 b_4(p, y)) \sin py + (b_1(p, y) - \varkappa_1 b_3(p, y)) \cos py + \\ &\quad + b_4(p, y) py \cos py - b_3(p, y) py \sin py \end{aligned}$$

$$\begin{aligned} b_1(p, y) &= \frac{\varkappa}{p} \int_{y_1}^{y} [\Psi(p, y) py \cos py + \Phi(p, y)(\varkappa_1 \cos py - py \sin py)] dy \\ b_2(p, y) &= \frac{\varkappa}{p} \int_{y_2}^{y} [-\Psi(p, y) py \sin py - \Phi(p, y)(\varkappa_1 \sin py + py \cos py)] dy \\ b_3(p, y) &= \frac{\varkappa}{p} \int_{y_1}^{y} [\Psi(p, y) \sin py + \Phi(p, y) \cos py] dy \\ b_4(p, y) &= \frac{\varkappa}{p} \int_{y_2}^{y} [-\Psi(p, y) \cos py + \Phi(p, y) \sin py] dy \\ \varkappa_1 &= \frac{\lambda + 3\mu}{\lambda + \mu}, \qquad \varkappa = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)} \end{aligned} \tag{1.12}$$

If the stresses in (1.5) are expressed in terms of displacements, and if one applies the Laplace transform, there results

$$-\lambda u(0,\pm 1)+(\lambda+2\mu)\frac{\partial V}{\partial y}\Big|_{y=\pm 1}+\lambda pU(p,\pm 1)=0$$
$$-\mu v(0,\pm 1)+\mu\frac{\partial U}{\partial y}\Big|_{y=\pm 1}+\mu pV(p,\pm 1)=0 \qquad (1.13)$$

Conditions (1.13) permit one to determine the $a_i(p)$ $(i=1,2,3,4)$ of (1.10).

2. Let us next investigate the skewsymmetric deformation of the semistrip. We assume that the functions $f_1(y)$ are odd in (1.1) to (1.4), and that the functions $f_2(y)$ are even, that the coefficients $a_2(p)$, $a_4(p)$ in the general solution are zero, and that in (1.12) the lower bounds of y_1 and y_2 are -1 and 0, respectively. Determining $a_1(p)$ and $a_3(p)$ with the aid of the condition (1.13), we have

$$a_1(p)=-\frac{1}{\varphi(p)}\Big[b_2(p,1)\Big(\cos^2 p-\frac{\lambda+2\mu}{\lambda+\mu}\Big)+pb_4(p;1)\Big(-p-\frac{\mu(\lambda+2\mu)}{(\lambda+\mu)^2}\frac{1}{p}\Big)-$$
$$-\frac{\lambda}{2\mu}u(0,1)\Big(\sin p+\frac{\mu}{\lambda+\mu}\frac{\cos p}{p}\Big)-\frac{1}{2}v(0,1)\Big(-\cos p+\frac{\lambda+2\mu}{\lambda+\mu}\frac{\sin p}{p}\Big)\Big] \qquad (2.1)$$

$$a_3(p)=-\frac{1}{\varphi(p)}\Big[-b_2(p,1)+pb_4(p,1)\Big(-\frac{\mu}{\lambda+\mu}\frac{1}{p}-\frac{\cos^2 p}{p}\Big)-$$
$$-\frac{\lambda}{2\mu}u(0,1)\frac{\cos p}{p}-\frac{1}{2}v(0,1)\frac{\sin p}{p}\Big],\qquad \varphi(p)=\sin p\cos p-p$$

The values of $b_2(p,1)$ and $b_4(p,1)$ are obtained from (1.12) by setting $y=1$.

From what has been said it follows that Expressions (1.10) for $U(p,y)$ and $V(p,y)$ contain the quantities $u(x,y)$, $\partial u/\partial x$, $\partial u/\partial y$, $v(x,y)$, $\partial v/\partial x$, $\partial v/\partial y$ when $x=0$. In the case of the boundary conditions (1.1) to (1.4), only some of these quantities are known. Hence, Expesssions (1.10) contain quantities known from the boundary conditions as well as unknown quantities.

In the plane of the complex variable p, the functions $U(p,y)$ and $V(p,y)$ have singularities at the points which correspond to the roots of Equation

$$\varphi(p)=\sin p\cos p-p=0 \qquad (2.2)$$

This equation has a root of the third order at the origin and has an infinite number of quadruples of complex roots of the first order

$$p_n,\ \bar{p}_n,\ -\bar{p}_n,\ -p_n \qquad (n=1,2,\ldots).$$

Correspondingly, $U(p,y)$ has a pole of the third order at the point $p=0$, and poles of the first order at the complex roots of Equation (2.2), while $V(p,y)$ has a pole of the fourth order at the point $p=0$ and poles of the first order at the remaining roots of Equation (2.2).

By hypothesis, $u(x,y)$ and $v(x,y)$ belong to the class of functions whose order of growth (as $x\to\infty$) is not higher than a power of x. Therefore, $U(p;y)$ and $V(p;y)$ must not have singularities to the right of the imaginary axis. For this it is necessary that the residues of $U(p;y)e^{px}$, $V(p;y)e^{px}$ at the poles p_n, $\bar{p}_n$ with a positive real part must be zero. But the vanishing of these residues is sufficient to insure that the growth order of $u(x,y)$, $v(x,y)$ (as $x\to\infty$) be not higher than a power of x.

Evaluating the residues of $U(p;y)e^{px}$ and $V(p;y)e^{px}$ at the pole p_n, we obtain

$$\mathrm{res}_{p_n}U(p;y)e^{px}=F(p_n)\frac{\varkappa}{p_n\varphi'(p_n)}\Big[\Big(\cos^2 p_n-\frac{\lambda+2\mu}{\lambda+\mu}\Big)\sin p_n y-p_n y\cos p_n y\Big]e^{p_n x}$$
$$\mathrm{res}_{p_n}V(p;y)e^{px}=F(p_n)\frac{\varkappa}{p_n\varphi'(p_n)}\Big[\Big(\cos^2 p_n+\frac{\mu}{\lambda+\mu}\Big)\cos p_n y+p_n y\sin p_n y\Big]e^{p_n x} \qquad (2.3)$$

where

$$F(p_n) = \int_0^1 \left\{ \Psi(p_n, y) \left[p_n y \sin p_n y + \cos p_n y \left(\cos^2 p_n + \frac{\mu}{\lambda + \mu} \right) \right] + \right.$$
$$\left. + \Phi(p_n, y) \left[p_n y \cos p_n y - \sin p_n y \left(\cos^2 p_n - \frac{\lambda + 2\mu}{\lambda + \mu} \right) \right] \right\} dy - \quad (2.4)$$
$$- \frac{\lambda(\lambda + 2\mu)}{\lambda + \mu} u(0, 1) \cos p_n - \frac{\mu(\lambda + 2\mu)}{\lambda + \mu} v(0, 1) \sin p_n$$

From the vanishing of the reidues of $U(p; y) e^{px}$, $V(p; y) e^{px}$ at the poles with positive real parts, it follows that

$$F(p_n) = 0 \qquad (n = 1, 2, \ldots) \quad (2.5)$$

Making use of the inversion theorem, we obtain

$$u(x, y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} U(p; y) e^{px} dp, \qquad v(x, y) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} V(p; y) e^{px} dp \quad (\sigma > 0)$$

Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} (\mathrm{res}_{-p_n} + \mathrm{res}_{-\bar{p}_n}) U(p; y) e^{px} + \mathrm{res}_0 U(p; y) e^{px}$$
$$v(x, y) = \sum_{n=1}^{\infty} (\mathrm{res}_{-p_n} + \mathrm{res}_{-\bar{p}_n}) V(p; y) e^{px} + \mathrm{res}_0 V(p; y) e^{px} \quad (2.6)$$

The residues at poles with negative real parts yield exponentially decaying terms; the residues at $p = 0$, yield terms which grow according to the power law.

In order that $u(x,y)$ and $v(x,y)$ may be decaying functions, it is necessary that the residues at the pole $p = 0$ vanish, i.e. it is necessary that

$$\mathrm{res}_0 U(p, y) e^{px} = 0, \qquad \mathrm{res}_0 V(p, y) e^{px} = 0 \quad (2.7)$$

But the conditions (2.7) are also sufficient that $u(x,y)$ and $v(x,y)$ be decaying functions; this follows from the fact that if these conditions are satisfied then $u(x,y)$ and $v(x,y)$ will not contain nondecaying terms.

Evaluating the residues of $U(p; y) e^{px}$ and $V(p; y) e^{px}$ at the point $p = 0$, we obtain

$$\mathrm{res}_0 U(p; y) e^{px} = -\frac{3y}{8} \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} \left(Ax^2 + 2Bx + \frac{\mu}{\lambda + 2\mu} C \right)$$
$$\mathrm{res}_0 V(p; y) e^{px} = \frac{3}{8} \frac{\lambda + 2\mu}{\mu(\lambda + \mu)} \left(\frac{1}{3} Ax^3 + Bx^2 + \frac{\mu}{\lambda + 2\mu} Cx + D \right) \quad (2.8)$$

where

$$A = \mu \left[\int_0^1 \frac{\partial v}{\partial x} \bigg|_{x=0} dy + u(0, 1) \right]$$

$$B = (\lambda + 2\mu) \int_0^1 \frac{\partial u}{\partial x} \bigg|_{x=0} dy - \lambda \int_0^1 v(0, y)\, dy + \lambda v(0, 1)$$

$$C = \lambda \int_0^1 \frac{\partial v}{\partial x} \bigg|_{x=0} y^2 dy + 2(3\lambda + 4\mu) \int_0^1 u(0, y)\, y dy + \lambda u(0, 1) \quad (2.9)$$

$$D = \frac{1}{3}(3\lambda + 4\mu) \int_0^1 \frac{\partial u}{\partial x} \bigg|_{x=0} y^3 dy - (3\lambda + 2\mu) \int_0^1 v(0, y)\, y^2 dy +$$
$$+ \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \int_0^1 v(0, y)\, dy + \frac{\lambda(3\lambda + 4\mu)}{3(\lambda + 2\mu)} v(0, 1)$$

From (2.7) and (2.8) it follows that

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 0 \tag{2.10}$$

Thus, if the conditions (2.5) and (2.10) are fulfilled, the displacements in the semistrip are decaying. From (2.6) and (2.7) it is seen that $u(x,y)$ and $v(x,y)$, when $x > 0$, $-1 \leqslant y \leqslant 1$, can be represented by uniformly convergent series, each term of which is exponentially decaying when $x \to \infty$. This implies that the stresses in the semistrip are also of a decaying character.

3. In (2.9), the expressions for A, B, C, D and in (2.4) those for $F(p_n)$, contain displacements and their derivatives when $x = 0$. One can transform these expressions so that they will contain only the quantities $u(0, y)$, $v(0, y)$, $\sigma_x(0, y)$ and $\tau_{xy}(0, y)$. Then the conditions (2.10) can be written in the form

$$\int_0^1 \tau_{xy}(0, y)\, dy = 0 \tag{3.1}$$

$$\int_0^1 \sigma_x(0, y)\, y dy = 0 \tag{3.2}$$

$$\frac{\lambda}{4(\lambda+\mu)} \int_0^1 \tau_{xy}(0, y)\, y^2 dy + 2\mu \int_0^1 u(0, y)\, y\, dy = 0 \tag{3.3}$$

$$\frac{3\lambda + 4\mu}{6(\lambda+\mu)} \int_0^1 \sigma_x(0, y)\, y^3 dy - 2\mu \int_0^1 v(0, y)(y^2 - 1)\, dy = 0 \tag{3.4}$$

and the conditions (2.5), in the form

$$\int_0^1 [\sigma_x(0, y)\, h_n(y) + \tau_{xy}(0, y)\, g_n(y) + 2\mu u(0, y)\, s_n(y) + 2\mu v(0, y)\, t_n(y)]\, dy = 0 \quad (n = 1, 2, \ldots) \tag{3.5}$$

where

$$\begin{aligned} h_n(y) &= p_n y \cos p_n y + \sin p_n y \left(\sin^2 p_n + \frac{\mu}{\lambda+\mu}\right) \\ g_n(y) &= p_n y \sin p_n y + \cos p_n y \left(\cos^2 p_n + \frac{\mu}{\lambda+\mu}\right) \\ s_n(y) &= p_n [p_n y \cos p_n y + \sin p_n y (\sin^2 p_n + 1)] \\ t_n(y) &= p_n [p_n y \sin p_n y - \cos p_n y \sin^2 p_n] \end{aligned} \tag{3.6}$$

The obtained system of conditions (3.1) to (3.5) make it possible to determine the unknown quantities which enter into the solution of the problem and they permit one to obtain two conditions which must be imposed on the boundary functions and which are necessary and sufficient for the existence of decaying solutions.

For the Problem 1, substituting (1.1) in (3.1) and (3.2), we obtain two conditions

$$\int_0^1 f_2(y)\, dy = 0, \qquad \int_0^1 f_1(y)\, y dy = 0 \tag{3.7}$$

The system of conditions (3.3) to (3.5) serves to determine the unknown quantities $u(0,y)$ and $v(0,y)$. The method of the determination of these quantities will be shown on an example of Problem 4.

For the Problem 2, substituting (1.2) into (3.1) and (3.3), we have the conditions

$$\int_0^1 f_2(y)\,dy = 0, \qquad \frac{\lambda}{4(\lambda+\mu)} \int_0^1 f_2(y)\, y^3\, dy + \int_0^1 f_1(y)\, y dy = 0 \tag{3.8}$$

The system of the remaining conditions (3.2), (3.4) and (3.5) permits one to determine $\sigma_x(0,y)$ and $v(0,y)$. However, for the derivation of the decaying solutions of the problem it is not necessary to determine $\sigma_x(0,y)$ and $v(0,y)$ from this system. Indeed, the solution of the problem contains the residues of $U(p;y)e^{px}$ and $V(p;y)e^{px}$ at the poles $-p_n, -\bar{p}_n$; the values of these residues can be obtained from (2.3) and (2.4) by replacing p_n by $-p_n$ and $-\bar{p}_n$, respectively. The unknown $\sigma_x(0,y)$ and $v(0,y)$ are contained in the expressions $F(-\bar{p}_n)$ and $F(-p_n)$ in such a way that they can easily be eliminated on the basis of conditions (3.5).

For Problem 3, substituting (1.3) into (3.2) and (3.4), we obtain two conditions

$$\int_0^1 f_1(y)\, y dy = 0, \qquad \frac{3\lambda+4\mu}{6(\lambda+\mu)} \int_0^1 f_1(y)\, y^3 dy - \int_0^1 f_2(y)\,(y^2-1)\, dy = 0 \tag{3.9}$$

Just as in the preceding case, it is not necessary to determine the unknown $\tau_{xy}(0,y)$ and $u(0,y)$ from the system of conditions (3.1), (3.3) and (3.5), for these unknowns are easily eliminated from the solution of the problem on the basis of conditions (3.5).

Let us now consider the Problem 4. The unknown quantities $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ are contained in all the conditions (3.1) to (3.4). Hence, the detrmination of these quantities is required not only for the construction of the solution of the problem, but also for the derivation of the conditions which are imposed on the boundary functions $f_1(y)$ and $f_2(y)$ in (1.4). The methods of the functional analysis permit us to determine $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ from the system of conditions (3.1), (3.2) and (3.5). After $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ have been determined by means of (3.3) and (3.4), we obtain (in a quite complicated form) the conditions which must be imposed on the boundary functions $f_1(y)$ and $f_2(y)$.

We note that the necessary and sufficient conditions (3.7) to (3.9) for the decaying of solutions of the Problems 1, 2 and 3, respectively, coincide with the conditions obtained for these problems in the paper [2] by different methods. In [2] there was established, however, only the sufficiency of the obtained conditions for Problems 2 and 3.

4. Let us consider the problem of determining the unknown quantities $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ from the system of conditions (3.1), (3.2) and (3.5). We note that this system of conditions can be supplemented with the conditions

$$F(\bar{p}_n) = 0 \qquad (n = 1, 2, \ldots) \tag{4.1}$$

which are equivalent to (3.5).

Making use of matrix notation [4], we may write

$$W(y) = \begin{pmatrix} \sigma_x(0,y) \\ \tau_{xy}(0,y) \end{pmatrix}, \qquad \Phi_n(y) = \begin{pmatrix} h_n(y) & g_n(y) \\ \bar{h}_n(y) & \bar{g}_n(y) \end{pmatrix}, \qquad \Phi_0(y) = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

$$A_n = \begin{pmatrix} a_n \\ \bar{a}_n \end{pmatrix}, \qquad A_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad a_n = -\int_0^1 [f_1(y)\, s_n(y) + f_2(y)\, t_n(y)]\, dy \tag{4.2}$$

For the matrix which is the integral of the product of two matrices $M_i(y)$ and $N_j(y)$, we introduce the notation

$$J(M_i, N_j) = \int_0^1 M_i(y)\, N_j(y)\, dy \tag{4.3}$$

Taking into account (1.4), we may represent the system of conditions (3.1), (3.2), (3.5) and (4.1) in the form

$$J(\Phi_n, W) = A_n \qquad (n = 0, 1, 2, \ldots) \tag{4.4}$$

The problem of the determination of the matrix $W(y)$ from the condition (4.4) is analogous to the problem of determining a function from its expansion in a nonorthogonal system of functions.

If the infinite system of matrices $\Phi_n(y)$ $(n = 0,1,2,\ldots)$ is complete, then the conditions (4.4) make it possible to detrmine the matrix $W(y)$ uniquely. The proof of the completeness of the infinite system of matrices $\Phi_n(y)$ $(n = 0,1,2,\ldots)$ has not been carried out. There is, however, reason to believe that this system is complete.

From the system of matrices $\Phi_n(y)$ $(n = 0,1,2,\ldots)$ we go over to an orthogonal system of matrices $\Psi_n(y)$ $(n = 0,1,2,\ldots)$. We say that a matrix $\Psi_n(y)$ is orthogonal to the matrix $\Psi_k(y)$ if

$$\int_0^1 \Psi_n(y)\,\Psi_k^*(y)\,dy = 0 \tag{4.5}$$

where the asterisk denotes the transposed matrix. We construct the system of matrices $\Psi_n(y)$ in the following way:

$$\begin{aligned}
&\Psi_0(y) = \Phi_0(y)\\
&\Psi_1(y) = C_0^{(1)}\Psi_0(y) + \Phi_1(y)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\Psi_n(y) = C_0^{(n)}\Psi_0(y) + C_1^{(n)}\Psi_1(y) + \ldots + C_{n-1}^{(n)}\Psi_n(y) + \Phi_n(y)\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{4.6}$$

Here the numerical square matrices $C_0^{(1)}, C_0^{(2)}, C_1^{(2)}, \ldots$ are selected in such a way that the matrix $\Psi_1(y)$ is orthogonal to the matrix $\Psi_0(y)$, the matrix $\Psi_2(y)$ is orthogonal to $\Psi_0(y)$ and to $\Psi_1(y)$, and so on. We shall write down the formulas for $C_0^{(n)}, C_1^{(n)}, \ldots, C_{n-1}^{(n)}$

$$\begin{aligned}
C_0^{(n)} &= -J(\Phi_n, \Psi_0^*)\,J^{-1}(\Psi_0, \Psi_0^*)\\
C_1^{(n)} &= -J(\Phi_n, \Psi_1^*)\,J^{-1}(\Psi_1, \Psi_1^*)\\
&\ldots\ldots\ldots\ldots\\
C_{n-1}^{(n)} &= -J(\Phi_n, \Psi_{n-1}^*)\,J^{-1}(\Psi_{n-1}, \Psi_{n-1}^*)
\end{aligned} \tag{4.7}$$

The existence of the inverses of the matrices in (4.7) is easily established on the basis of the linear independence of the matrices $\Phi_n(y)$. By construction, the system of matrices $\Psi_n(y)$ $(n = 0,1,2,\ldots)$ is such that $\Psi_n(y)$ is orthogonal to all $\Psi_k(y)$ for which $k < n$. Recalling that the transpose of a product of martices is equal to the product of the transposed factors taken in reverse order, we find that the matrix $\Psi_n(y)$ is orthogonal to all matrices $\Psi_k(y)$ of the system (4.6).

Let us expand $W(y)$ into a series by means of the orthogonal system of matrices $\Psi_n(y)$

$$W(y) = \sum_{n=0}^{\infty} \Psi_n^*(y)\,\alpha_n \tag{4.8}$$

For the determination of the matrices α_n, which are the coefficients of the series (4.8), we multiply both parts of Equation (4.8) from the left by $\Psi_m(y)$ and integrate with respect to y from 0 to 1. Since the system $\Psi_n(y)$ is orthogonal, we get only one term with α_m on the right, i.e.

$$J(\Psi_m, W) = J(\Psi_m, \Psi_m^*)\,\alpha_m$$

Multiplying this relation from the left by a matrix which is the coefficient of α_m, i.e. by $J^{-1}(\Psi_m, \Psi_m^*)$, we obtain α_m. Hence, for the coefficients of (4.8) we have

$$\alpha_n = J^{-1}(\Psi_n, \Psi_n^*)\,J(\Psi_n, W) \tag{4.9}$$

The coefficients α_n can be expressed in terms of A_n by the use of the relation (4.4). For this purpose we consider successively, beginning with $n = 0$, the matrices $J(\Psi_n, W)$. With the aid of (4.6) and (4.4) we obtain

$$J(\Psi_0, W) = A_0 = B_0$$
$$J(\Psi_1, W) = C_0^{(1)}B_0 + A_1 = B_1$$
$$J(\Psi_2, W) = C_0^{(2)}B_0 + C_1^{(2)} B_1 + A_2 = B_2 \qquad (4.10)$$
$$\cdots\cdots\cdots\cdots$$

Substituting these expressions in (4.9), we obtain the values of α_n. Thus one can determine the terms of the series (4.8) for $W(y)$ successively.

5. Let us consider a symmetric deformation of a semistrip. In this case one has to assume that in (1.1) to (1.4) the functions $f_1(y)$ are even and the functions $f_2(y)$ are odd, that the coefficients $a_1(p)$, $a_3(p)$ are zero in the general solution (1.10), and that in (1.12) the lower bounds of y_1 and y_2 are 0 and -1, respectively. Determining $a_2(p)$ and $a_4(p)$ from (1.13) we have

$$a_2(p) = \frac{1}{\varphi(p)}\Big\{b_1(p, 1)\left(\frac{\mu}{\lambda+\mu} + \cos^2 p\right) - b_3(p, 1)\left(\frac{\mu(\lambda+2\mu)}{(\lambda+\mu)^2} + p^2\right) -$$
$$- \frac{\lambda}{2\mu} u(0, 1)\left(\cos p - \frac{\mu}{\lambda+\mu}\frac{\sin p}{p}\right) + \frac{1}{2} v(0, 1)\left(-\sin p - \frac{\lambda+2\mu}{\lambda+\mu}\frac{\cos p}{p}\right)\Big\}$$
$$a_4(p) = \frac{1}{\varphi(p)}\Big\{- b_1(p, 1) + \left(\frac{\mu}{\lambda+\mu} + \sin^2 p\right) b_3(p, 1) - \qquad (5.1)$$
$$- \frac{\lambda}{2\mu} u(0, 1)\frac{\sin p}{p} + \frac{1}{2} v(0, 1)\frac{\cos p}{p}\Big\}$$
$$(\varphi(p) = \sin p \cos p + p)$$

The values of $b_1(p, 1)$ and $b_3(p, 1)$ are obtained from (1.12) if one takes $y = 1$.

The functions $U(p,y)$ and $V(p,y)$ have singularities in the complex plane at he points which correspond to the roots of Equation

$$\varphi(p) = \sin p \cos p + p = 0 \qquad (5.2)$$

Equation (5.2) has a first order root at the origin and has an infinite number of quadruples of complex roots of the first order p_n, $\bar{p}_n$, $-\bar{p}_n$, $-p_n$ $(n = 1, 2, \ldots)$. It is easy to see that $U(p; y)$ has a second order pole at the point $p = 0$, and first order poles at the complex roots of Equation (5.2), while $V(p;y)$ has first order poles at all the roots of Equation (5.2).

Equating to zero the residues of $U(p, y)\,e^{px}$ and $V(p, y)\,e^{px}$ at the poles p_n and $\bar{p}_n$ with positive real parts, we obtain a system of conditions which are necessary and sufficient in order that the growths of $u(x,y)$ and $v(x,y)$ be not more than a power of x, as x tends to infinity. This system of conditions can be written in the form

$$\int_0^1 [\sigma_x(0, y)\,h_n(y) + \tau_{xy}(0, y)\,g_n(y) + 2\mu u(0, y)\,s_n(y) + 2\mu v(0, y)\,t_n(y)]\,dy = 0 \qquad (5.3)$$
$$(n = 1, 2, \ldots)$$

where

$$h_n(y) = p_n y \sin p_n y - \left(\cos^2 p_n + \frac{\mu}{\lambda+\mu}\right)\cos p_n y$$
$$g_n(y) = - p_n y \cos p_n y + \left(\sin^2 p_n + \frac{\mu}{\lambda+\mu}\right)\sin p_n y \qquad (5.4)$$
$$s_n(y) = p_n\,[p_n y \sin p_n y - (1 + \cos^2 p_n)\cos p_n y]$$
$$t_n(y) = p_n\,[- p_n y \cos p_n y - \cos^2 p_n \sin p_n y]$$

Equating to zero the residues of $U(p, y)e^{px}$ and $V(p, y)e^{px}$ at the pole $p = 0$, we obtain two more conditions which together with (5.3) are necessary and sufficient that $u(x,y)$ and $v(x,y)$ may not contain nondecaying terms. These conditions have the form

$$\int_0^1 \sigma_x(0, y)\,dy = 0 \tag{5.5}$$

$$2\mu\int_0^1 u(0, y)\,dy + \frac{\lambda}{2(\lambda+\mu)}\int_0^1 \tau_{xy}(0, y)\,y\,dy = 0 \tag{5.6}$$

The system of conditions (5.3), (5.5) and (5.6) make it possible to determine the unknown quantities contained in the solution of the problem, and to obtain one condition which must be imposed upon the boundary functions to get necessary and sufficient conditions for the existence of a decaying solution. Let us consider separately each of the four problems which correspond to the conditions (1.1) to (1.4). For the Problem 1, substituting (1.1) into (5.5), we obtain the condition

$$\int_0^1 f_1(y)\,dy = 0 \tag{5.7}$$

The conditions (5.3) and (5.6) permit one to determine the unknown quantities $u(0,y)$ and $v(0,y)$, the knowledge of which is necessary for obtaining a decaying solution of the problem. In order to go over to the matrix form, and in order to be able to make use of the method presented above, we replace the integral from 0 to 1 in the conditions (5.3) and (5.6) by integrals from -1 to 1, and we also add one obvious condition

$$\int_{-1}^1 v(0, y)\,dy = 0$$

For the Problem 2, substituting (1.2) into (5.6), we obtain the condition

$$\int_0^1 f_1(y)\,dy + \frac{\lambda}{2(\lambda+\mu)}\int_0^1 f_2(y)\,y\,dy = 0 \tag{5.8}$$

It is not necessary to determine the unknown $\sigma_x(0,y)$, $v(0,y)$ from the conditions (5.3) and (5.5) because these quantities enter into the solution in such a way that they can easily be eliminated from it on the basis of the relation (5.3).

For the Problem 3, we substitute (1.3) into (5.5), and obtain a condition on the function $f_1(y)$. We not that this condition coincides with the condition (5.7) for the Problem 1. The system of conditions (5.3) permits one to eliminate the unknown $\tau_{xy}(0,y)$ and $u(0,y)$ which enter the solution of the problem

In case of Problem 4, the unknown quantities $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ appear in all the conditions (5.3), (5.5) and (5.6). The determination of these quantities is needed for the construction of the solution of the problem as well as for the derivation of the condition which has to be imposed on the boundary functions $f_1(y)$ and $f_2(y)$ in (1.4).

For the determination of $\sigma_x(0,y)$ and $\tau_{xy}(0,y)$ in the conditions (5.3) and (5.5), we replace the integrals from 0 to 1 by integrals from -1 to 1, we add still another obvious condition

$$\int_{-1}^1 \tau_{xy}(0, y)\,dy = 0$$

go over to the matrix notation, and use the method presented above in the solution of Problem 4 for the case of skew-symmetric deformations.

Substituting the quantity $\tau_{xy}(0,y)$ thus obtained into (5.6), and replacing $2\mu u(0,y)$ by $f_1(y)$, we obtain the condition which must be imposed on the boundary functions $f_1(y)$ and $f_2(y)$ to yield necessary and sufficient conditions for the existence of a decaying solution.

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